Normal mode oscillations of a nonlocal composite matter wave soliton

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The existence of stable bound states of three solitons in a Bose-Einstein condensate with nonlocal interactions is demonstrated by means of the variational approach (VA) and numerical simulations. The potential of interaction between solitons derived from VA is shown to be of molecular type, i.e., attractive at long distances and repulsive at short distances. Normal modes of a three-soliton molecule are investigated by computing small amplitude oscillations of individual solitons near their equilibrium positions. Symmetric and asymmetric stretched states of the molecule are prepared and used as initial conditions in numerical simulations of the nonlocal Gross-Pitaevskii equation. As opposed to usual triatomic molecules, we find that the frequency of the asymmetric mode of a three-soliton molecule is smaller than the one of the symmetric mode. Possible experimental settings for the observation of these results are briefly discussed.

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I. INTRODUCTION

The interaction between solitons and formation of their bound states has been the subject of long-standing interest in the physics of nonlinear waves. Some features of soliton interactions, which were theoretically predicted long ago, are finding confirmation in present-day experiments [1,2]. The experimental demonstration of stable two- and three-soliton complexes, so-called *soliton molecules*, in dispersion-managed optical fibers [3–5] and revealing their possibilities for advanced optical telecommunications [6], has been notable progress in this direction. Soliton interactions are important from the viewpoints of both fundamental physics and practical applications. Motivation for optical communications has led to the discovery of soliton interactions in fibers in early research on optical solitons [7]. For example, it is known that in soliton-based fiber-optic communication lines [8–10] the interaction of copropagating solitons can reduce the overall performance of the system.

Another physical medium, where solitons can exist, is the Bose-Einstein condensate (BEC) of a diluted atomic gas. Experimental and theoretical research on solitons in BEC has been reported in many publications (see review articles [11–13]). The majority of papers are devoted to properties of single solitons and soliton trains. Evidence on the interactions between matter-wave solitons was inferred from the behavior of neighboring solitons, oscillating in a quasi-one-dimensional (quasi-1D) harmonic trap [2,14]. Collective dynamics of a chain of solitons, confined by external potential, in the adiabatic limit has been investigated in Ref. [15]. Similar phenomena in two-component BEC were studied in Ref. [16]. Regimes to produce bound states of matter-wave solitons from their collisions were found in Ref. [17]. It should be noted that systematic investigation of the interaction process of just two or three solitons in BEC requires precise production and manipulation techniques, which is being developed nowadays [18–20].

An important fact to be stressed here is that in the mean-field description of BEC, in terms of the Gross-Pitaevskii equation (GPE) with usual contact atomic interactions, solitons cannot form stable bound states with finite binding energy. The interaction force between them depends on the phase difference and can be either attractive or repulsive, and their interaction potential is not of molecular type. Potential curves for two colliding nonlinear Schrödinger solitons were calculated in Ref. [21]. Soliton complexes in this model, therefore, do not feature a fixed equilibrium distance, analogous to the bond length of atomic molecules. In this respect, it is worth mentioning that a breather consisting of two equal and in-phase solitons, periodically passing through each other, as predicted by standard nonlinear Schrödinger equation (NLSE) with focusing cubic nonlinearity, has not been found in experiments [7]. The reason is that when the two solitons merge, higher order nonlinear phenomena come into play, which are not captured by the standard NLSE.

The situation is different in BEC with long-range dipole-dipole atomic interactions. In qualitative terms, one can say that in dipolar BEC the atoms within one soliton can directly interact with atoms inside another soliton, so that combined dipolar and usual phase-dependent interactions of solitons may open the way toward formation of true matter-wave soliton molecules. The existence of stable bound states of bright matter-wave solitons in dipolar BEC, where solitons attract each other at long distances and repel at short distances, has been theoretically predicted in several papers. Specifically, soliton bound states in a stack of quasi-1D and quasi-2D traps were reported in Refs. [22] and [23], respectively. In these models, individual solitons, forming the bound state, reside in separate stacks. Existence of bright solitons and dark-soliton pairs in a dipolar Tonks-Girardeau gas was investigated in Ref. [24]. Numerical analysis of soliton bound states in quasi-2D and 3D dipolar BEC were also reported in Refs. [25,26]. Formation of bound states of solitons and

their fusion resulting from collision of dipolar solitons have been investigated in Ref. [27]. The vibration spectrum of a two-soliton molecule in dipolar BEC, confined to a single quasi-1D trap, was studied in Ref. [28], while the potential of interaction, formation of two-soliton molecules, and their binding energy in one-dimensional dipolar BEC were studied by variational approach and numerical simulations in Ref. [29]. Dark solitons in dipolar BEC, interacting with each other via molecular-type potentials and capable of forming stable bound states, were recently reported in Refs. [30,31]. Soliton bound states and clusters in nonlocal optical media are also intensively investigated (for a recent review, see the book [32]). Once a soliton molecule has been created, many interesting phenomena, similar to those observed in molecular physics, can be modeled with them.

Our main objective in this paper is to study the dynamics and normal mode oscillations of three-soliton molecules, which can exist in nonlocal media. To this end, we develop a variational approach (VA) [33,34] to find the stationary shape of a three-soliton molecule, reveal the character of the interaction potential, and estimate the frequency of small-amplitude oscillations of solitons near their equilibrium positions. VA stationary profiles of three-soliton molecules are also found in very good agreement with the numerical ones obtained from a self-consistent (SC) procedure [35] applied to the GPE. To explore the molecular dynamics, we prepare symmetric and asymmetric stretched states of the molecule by imposing constant and nonuniform chirping of the ground-state wave function, using them as initial conditions for the GPE, and recording the positions of each soliton during their time evolution, which constitute the basis of our numerical experiments.

As a result, we show that when considered in proper coordinates the GPE dynamics simplifies, displaying harmonic oscillations which resemble the ones of normal modes of usual triatomic linear molecules. In contrast to what is observed in molecular physics, however, we find that the oscillation frequency of the motion induced from a symmetric stretching is always larger than the one induced from an asymmetric stretching of the three-soliton molecule. We find that the VA predictions for stationary three-soliton molecules and for the symmetric oscillations of the molecule are in excellent agreement with numerical GPE integrations. The VA, however, does not allow us to make predictions for asymmetric oscillations due to the difficulty of finding suitable trial functions for this case. Normal mode oscillations have also been investigated for topological soliton bound states of the sine-Gordon equation [36] and for the displaced dynamics of binary BEC mixtures [37].

The paper is organized as follows. In Sec. II, we develop the VA using the Gauss-Hermite trial function for a three-soliton molecule and check its validity by comparing its predictions with the results of numerical solution of the governing nonlocal GPE. In Sec. III, we consider initially deformed states suitable to excite internal mode oscillations of the molecule and use them as initial conditions for numerical integrations of the GPE. Results are then compared with predictions of the VA analysis. In Sec. IV, we briefly summarize our findings and discuss the generality of our results with respect to other types of nonlocal interactions. Possible experimental settings and areas of research where the obtained results might be useful are also briefly discussed.

II. MODEL EQUATIONS AND VARIATIONAL ANALYSIS

The governing equation of our model is a 1D nonlocal Gross-Pitaevskii equation, represented in normalized units as follows:

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+q|\psi|^2\psi + g\psi\int_{-\infty}^{+\infty}R(|x-\xi|)\,|\psi(\xi,t)|^2 d\xi=0, \tag{1}$$

where $\psi(x,t)$ is the mean field wave function of the condensate and q and g are coefficients of nonlinearity, responsible for the local contact and long-range nonlocal atomic interactions, respectively. The wave function is normalized to the number of atoms in the condensate $N=\int_{-\infty}^{+\infty}|\psi(x)|^2dx$, which is a conserved quantity of Eq. (1). Since the nonlocal interaction is essential for the existence of soliton molecules and molecular dynamics, we shall concentrate mainly on the case $q=0$ and discuss at the end that results may be preserved also in the presence of contact interactions. We also remark that in experiments it is possible to detune the cubic nonlinearity to zero by means of Feschbach resonances [38].

The response function $R(x)$ in Eq. (1) characterizes the degree of nonlocality of the medium, which shows how strongly the properties at a given location depend on the properties of its neighborhood. For analytical convenience, we consider a Gaussian function normalized to one

$$R(x)=\frac{1}{\sqrt{2\pi}w}\exp\left(-\frac{x^2}{2w^2}\right) \tag{2}$$

and show in the last section that similar results can be obtained also for long-ranged response functions with algebraic, instead of exponential, decay at large distances. The parameter w in Eq. (2) designates the strength of nonlocality. At $w\to 0$ the response function resembles the Dirac δ function. In this case, the medium is called weakly nonlocal. In the opposite case of large w, compared to the waist of the excitation, the medium is called highly nonlocal. The response function for a dipolar BEC, confined to quasi-1D trap, was derived in Ref. [39].

For three-soliton bound states, we can employ the variational approach similar to that developed in Refs. [28,29]. As a suitable trial function, we use the second Gauss-Hermite function

$$\psi(x,t)=A\left(2\frac{x^2}{a^2}-1\right)\exp\left[-\frac{x^2}{2a^2}+ibx^2+i\phi\right], \tag{3}$$

where the variational parameters $A(t), a(t), b(t), \phi(t)$ mean amplitude, width, chirp, and phase, respectively. It should be noted that this waveform can be modeled by three Gaussian functions, arranged in antiphase configuration. When the phase difference between adjacent solitons differs from $\phi=\pi$, stable bound state of three solitons does not emerge, as we have found from numerical simulations.

The norm of the trial function, which is proportional to reduced number of atoms, is $N=2A^2a\sqrt{\pi}$. To develop the

VA, we note that Eq. (1) can be obtained from the Lagrangian density:

$$\mathcal{L} = \frac{i}{2}(\psi\psi_t^* - \psi^*\psi_t) + \frac{1}{2}|\psi_x|^2 - \frac{1}{2}q|\psi|^4 - \frac{1}{2}g|\psi(x,t)|^2\int_{-\infty}^{\infty} R(x-\xi)|\psi(\xi,t)|^2 d\xi. \quad (4)$$

Using the response function (2) and the ansatz in Eq. (3), we evaluate the Lagrangian density (4). Subsequent integration over the space variable $L = \int \mathcal{L} dx$ yields the averaged Lagrangian

$$\frac{L}{N} = \frac{5}{2}a^2 b_t + \phi_t + \frac{5}{4a^2} + 5a^2b^2 - \frac{41qN}{128\sqrt{2\pi}a} - \frac{gN}{2\sqrt{2\pi}}F(a,w), \quad (5)$$

where

$$F(a,w) = \frac{w^8 + 2w^6a^2 + \frac{15}{4}w^4a^4 + \frac{7}{4}w^2a^6 + \frac{41}{64}a^8}{(w^2+a^2)^{\frac{9}{2}}}. \quad (6)$$

From the Euler-Lagrange equations, $d/dt(\partial L/\partial v_t) - \partial L/\partial v = 0$ for the variational parameters $v \to a,\ b,\ \phi$, we obtain the following equation for $a(t)$:

$$a_{tt} = \frac{1}{a^3} - \frac{41\,q\,N}{320\sqrt{2\pi}a^2} + \frac{g\,N}{5\sqrt{2\pi}}\frac{\partial F(a,w)}{\partial a}. \quad (7)$$

This equation has formal analogy with the equation of motion for a unit mass particle performing oscillations in the anharmonic potential $U(a)$:

$$U(a) = \frac{1}{2a^2} - \frac{41\,q\,N}{320\sqrt{2\pi}\,a} - \frac{g\,N}{5\sqrt{2\pi}}F(a,w), \quad (8)$$

depicted in the top panel of Fig. 1. The minimum of the potential (8) at $a = a_0$ corresponds to stationary width of the molecule. The frequency of small-amplitude oscillations of the molecule can be estimated from $\omega_0^2 = \partial^2 U/\partial a^2|_{a\to a_0}$. It should be pointed out that the interaction potential between solitons, given by Eq. (8), is of a molecular type; i.e., solitons attract each other at long distance ($\partial U/\partial a|_{a>a_0} > 0$) and repel at short distance ($\partial U/\partial a|_{a<a_0} < 0$), so that if $a > a_0$ the distance between solitons tends to shrink and for $a < a_0$ it tends to expand. At the equilibrium distance, attractive and repulsive forces balance each other, and the solitons remain motionless. In the bottom panel of Fig. 1, we show the stationary wave profile, found from the fixed point of Eq. (7), and compare it with the exact wave profile numerically obtained from a self-consistent (SC) procedure [35] applied to GPE (1). The excellent agreement confirms the validity of the trial function in Eq. (3) for our analytical calculations.

In analogy with the bond length of ordinary molecules composed of neutral atoms, the distance between maxima of two lateral solitons $\Delta = 2x_m = \sqrt{10}\,a$ can be a characteristic parameter of the soliton molecule.

To check the accuracy of the VA, we have periodically modulated in time the strength of the nonlocal nonlinearity, $g(t)$, and compared the results of Eq. (7) with numerical solution of the GPE (1). In experiments with dipolar BEC, such a *dipolar nonlinearity management* can be implemented by means of rotating magnetic fields [40,41]. Alternatively,

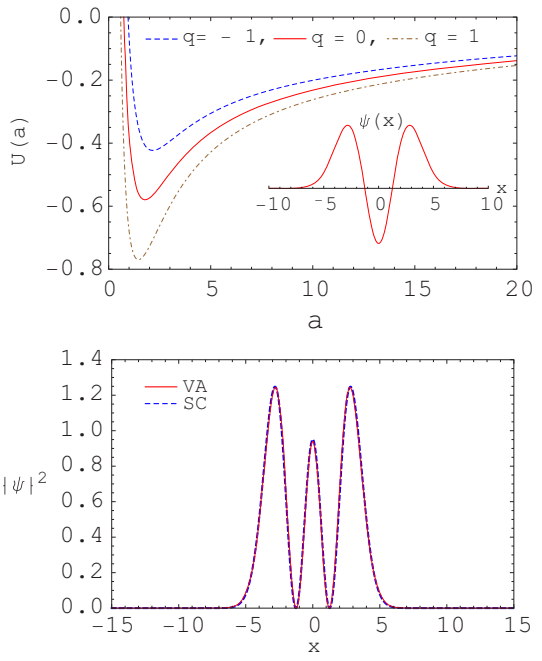

FIG. 1. Top panel: The potential $U(a)$ for $g = 10$ and $q = 0, \pm 1$. The inset shows the wave function of a three-soliton molecule for $q = 0$. Bottom panel: The modulo square of the wave function, according to VA for parameter values $N = 6$, $w = 5$, $q = 0$, $g = 10$, $A = 0.975$, and $a = 1.781$. Dashed line represents the stationary wave profile, constructed using the self-consistent procedure [35]. The two curves nearly coincide, which shows that Eq. (3) represents a good trial function.

this can be achieved by slowly varying the polarization angle θ, since $g \sim (1 - 3\cos^2\theta)$, where θ is the angle between the long axis of the quasi-1D trap and the dipoles. Figure 2 illustrates the dynamics of the three-soliton molecule under

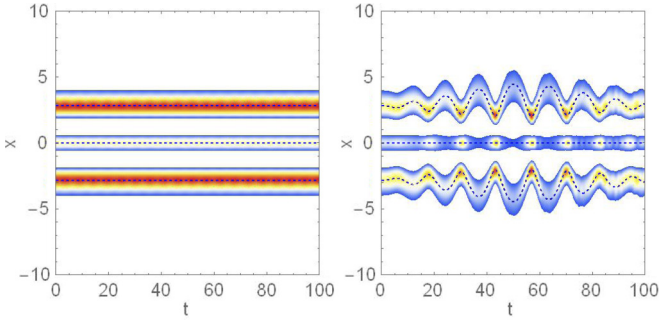

FIG. 2. Left panel: Stable propagation of the three-soliton molecule with parameters predicted by VA. The density plot is obtained by numerical solution of the GPE (1). Dashed lines correspond to positions of maxima of the central and lateral solitons $x_m = \pm\sqrt{5/2}\,a(t)$, where the time-dependent parameter $a(t)$ is evaluated from Eq. (7). Right panel: Periodic variation of the strength of dipolar interactions $g(t) = g_0[1 + \epsilon\sin(\omega_0 t)]$ at resonant frequency $\omega_0 = 0.529$ gives rise to vibration of lateral solitons with growing amplitude, while the central soliton remains at origin due to the symmetry. Parameter values: $g_0 = 10$, $\epsilon = 0.1$. Other parameters are similar to Fig. 1.

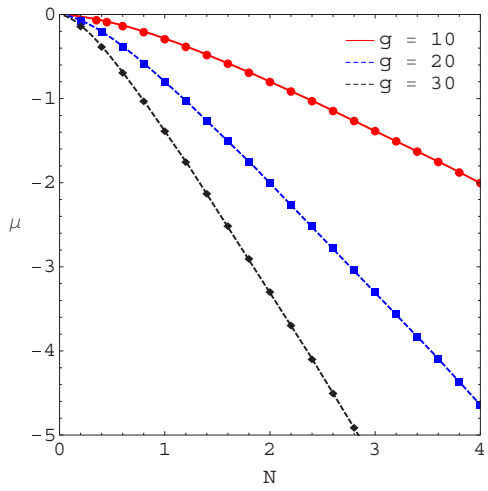

FIG. 3. The chemical potential as a function of the norm for three values of the nonlocal coefficient and $q = 0$. The curves are drawn according to VA Eqs. (11) and (12), while the symbols represent the data, obtained by SC procedure [35]. It is evident that stronger nonlocal interaction leads to more stable soliton molecules.

varying strength of nonlocal interaction. As can be seen from this figure, the VA provides an accurate description of the dynamics. The stability of localized solutions for nonlinear wave equations can be examined by means of the Vakhitov-Kolokolov (VK) criterion [42]. Following the usual procedure [34], we look for stationary solutions of the GPE as $\psi(x,t) = \varphi(x)\exp(-i\mu t)$, where μ denotes the chemical potential. The time-independent GPE takes the form

$$\mu\varphi + \frac{1}{2}\varphi_{xx} + q\varphi^3 + g\varphi\int_{-\infty}^{\infty} R(|x-z|)\varphi^2(z)dz = 0, \quad (9)$$

and the corresponding Lagrangian density is

$$\mathcal{L} = \frac{1}{4}\left[\varphi_x^2 - 2\mu\varphi^2 - q\varphi^4 - g\varphi^2\int_{-\infty}^{\infty} R(|x-z|)\varphi^2(z)dz\right].$$

Performing further standard VA procedures with the ansatz

$$\varphi(x) = A\left(2\frac{x^2}{a^2} - 1\right)\exp\left(-\frac{x^2}{2a^2}\right), \quad (10)$$

and using the response function (2), we get the following expressions for the chemical potential and norm:

$$\mu = -\frac{q\,N}{a}\frac{123}{256\sqrt{2\pi}} - \frac{g\,N}{\sqrt{2\pi}}\left(F + \frac{a}{4}\frac{\partial F}{\partial a}\right), \quad (11)$$

$$N = \frac{320\sqrt{2\pi}}{a\left(41q - 64\,g\,a^2\frac{\partial F}{\partial a}\right)}, \quad (12)$$

with the function $F(a,w)$ given by Eq. (6). From the parametric plot $\mu(a)$ versus $N(a)$ depicted in Fig. 3, one can see that the condition $\frac{d\mu}{dN} < 0$ is always satisfied, which suggests, according to VK criterion, the stability of the three-soliton molecule for different values of the coefficient g. As expected, the stronger attraction between solitons leads to more stability of the molecule.

III. NUMERICAL RESULTS

To explore the molecular three-soliton dynamics, we need to prepare initial symmetric and asymmetric stretched states of the molecule and use them as initial conditions for numerical simulations of the GPE. However, stretching and releasing the molecule in such a way that each soliton oscillates near its equilibrium position, while the center of mass of the molecule remains at rest (as usually presumed by the normal modes theory), is a quite challenging problem. That is why we employ another approach to excite symmetric and asymmetric modes of the molecule, initially prepared in its ground state ψ_{gs}. In particular, to excite only the symmetric mode of the molecule, when the flanking solitons oscillate in antiphase, while the central soliton does not move, we impose constant chirping $\psi_{gs}e^{ibx^2}$ with a small chirp parameter $b \ll 1$, $x \in [-\infty,\infty]$. To excite both the symmetric and asymmetric modes, we impose inhomogeneous chirping $b = 0$ for $x < 0$, and $b \neq 0$ for $x > 0$. It should be stressed that inhomogeneous chirping induces vibration of all solitons, as well as motion of the entire molecule, as shown in the right panel of Fig. 4. Below we employ the reference frame, attached to the moving molecule. In Fig. 5, we show the time evolution of the center-of-mass positions of individual solitons of a three-soliton molecule, excited as described above.

Since the resulting dynamics of the molecule is a superposition of different modes, its periodic character is not readily recognized in the actual center-of-mass coordinates x_i (see the lower left panel of Fig. 5). The periodic motions, however, become evident if one introduces the coordinates η_i defined as

$$\eta_1(t) = x_1(t) - x_3(t), \quad \eta_2(t) = x_1(t) + x_3(t), \quad (13)$$

where $x_i(t)$ denote the displacement of the solitons with respect to their equilibrium positions. Note that apart from constant factors, these coordinates are just the same as the normal mode coordinates of usual linear triatomic molecules. Obviously, the model is valid for small-amplitude oscillations of solitons when anharmonic effects are negligible.

In normal mode coordinates (13), the dynamics indeed looks periodic, and the frequencies of the symmetric and asymmetric modes are easily identified (see the right panels

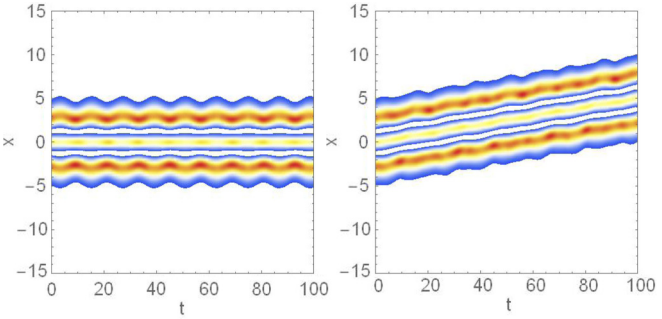

FIG. 4. Excitation of the symmetric (left) and both the symmetric and asymmetric (right) modes of the three-soliton molecule. The symmetric mode has been excited via constant chirping $\psi_{gs}e^{ibx^2}$ with $b = 0.02$. For the asymmetric mode $b = 0$ for $x \leqslant 0$ and $b = 0.02$ for $x > 0$.

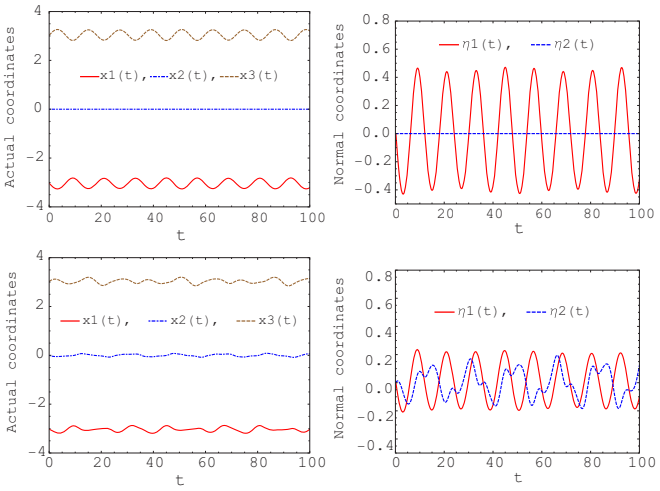

FIG. 5. Center-of-mass positions of individual solitons of a three-soliton molecule, represented using actual (left panels) and normal mode coordinates (right panels), according to numerical solution of Eq. (1), in the reference frame, attached to the moving molecule. Small deviations from pure sinusoidal character in the asymmetric mode (blue dashed line) is due to the matter exchange between solitons. The normal mode frequencies found from GPE simulations are equal to $\omega_s = 0.53$, $\omega_a = 0.35$ for parameter values fixed as in Fig. 1.

of Fig. 5). In this respect, the soliton molecule behaves similar to the usual triatomic molecule. However, there is also a significant difference between these two systems. It concerns the flow of matter between solitons during the time evolution, which is considered below.

The mass of each soliton ($m_i, i = 1, 2, 3$) is proportional to its norm

$$m_1(t) = \int_{-\infty}^{z_1(t)} ndx,\ m_2(t) = \int_{z_1(t)}^{z_2(t)} ndx,\ m_3(t) = \int_{z_2(t)}^{\infty} ndx,$$

where $n = |\psi(x,t)|^2$ is the density of the condensate according to GPE (1) and $z_1(t)$, $z_2(t)$ are the left and right borders of the middle soliton (where the field amplitude vanishes $\psi(x,t) \to 0$).

Time dependence of these quantities implies that each soliton of the molecule periodically expands, shrinks, and moves. Evaluation of masses of solitons according to above formulas shows that there is small exchange of matter between solitons, when the vibrations of the molecule has been excited, as illustrated in Fig. 6. The strength of interaction between solitons, and therefore vibration frequency of soliton bound states, depends on the number of atoms (expressed via norm).

The frequency of symmetric oscillations of the molecule can be predicted by VA through the second derivative of the potential in Eq. (8),

$$\omega_s = \sqrt{\partial^2 U/\partial a^2|_{a\to a_0}}. \tag{14}$$

We find that this expression leads to results that are in very good agreement with GPE numerical calculations, as shown below. For the frequency of asymmetric mode ω_a, however, analytic estimate is not available.

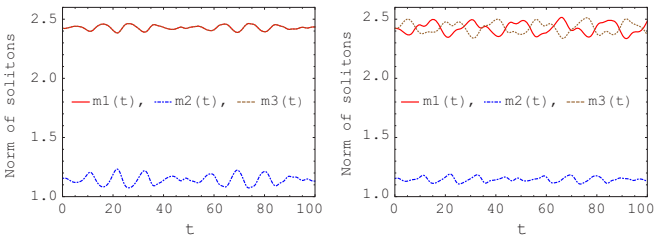

FIG. 6. Excitation of the molecule's vibrational modes leads to small exchange of matter between solitons. The total mass is always conserved, $m_1(t) + m_2(t) + m_3(t) = \text{const}$. When the symmetric mode has been excited (left), flanking solitons exchange equal amount of matter with the central soliton. In this case, the curves for $m_1(t)$ and $m_3(t)$ coincide. For the asymmetric mode (right), there is a dynamic imbalance between masses of flanking solitons $m_1(t)$ and $m_3(t)$.

Quite interestingly, we find that the frequency of the asymmetric mode is always smaller than the one of the symmetric mode

$$\omega_s > \omega_a. \tag{15}$$

It is well known that in usual triatomic molecules the opposite relation holds. In Fig. 7, the symmetric and asymmetric mode frequencies are plotted as a function of the norm. From this figure, it is also evident that the numerical results for the symmetric frequency are in excellent agreement with the ones derived from the VA expression in Eq. (14). Similar behaviors were found for generic parameter values and for other initial conditions. It is not simple, however, to account for the asymmetric oscillation frequency of the molecule by means of the VA. In this respect, notice that the ansatz in Eq. (3) does not allow any asymmetric dynamics.

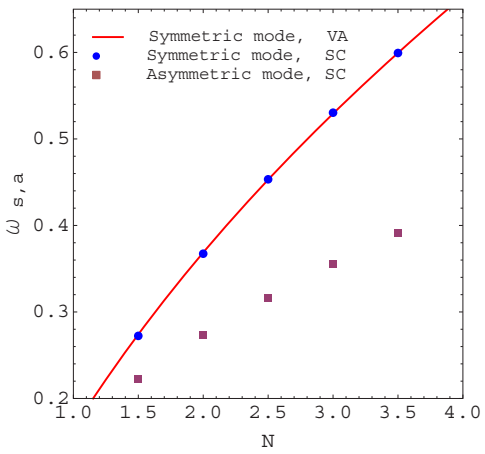

FIG. 7. Normal mode frequencies of the three-soliton molecule, obtained by numerical solution of the GPE (1) with initial wave forms, corresponding to different norms of the molecule. Validity of the relation (15) is confirmed for all selected parameters. The symmetric mode frequency is predicted by the VA expression in Eq. (14) (red solid line), while for the asymmetric mode frequency an analytic estimate is not available.

IV. DISCUSSION AND CONCLUSIONS

Before closing this paper, we feel compelled to discuss in more detail the feasibility of the above results for dipolar BEC and possible experimental settings to verify the proposed model. In this regard, the following remarks are in order.

A. About the response functions

First, for analytical convenience we have employed a normalized Gaussian function for the kernel $R(x)$ in Eq. (1) which does not possess the required long-ranged algebraic decay $\sim 1/x^3$ typical of dipolar interactions. On the other hand, an expression for the dipolar response function for the one-dimensional setting was derived in Ref. [39], involving the special functions. This kernel, however, appears to be quite complicated for analytical considerations. A more convenient kernel was proposed by introducing a cutoff parameter δ [43]

$$R(x) = \frac{\delta^3}{(x^2+\delta^2)^{3/2}}\,. \tag{16}$$

Equation (16) correctly describes the asymptotic behavior of dipolar forces, decaying at long distances as $\sim 1/x^3$, and unlike the response function of Ref. [39], does not feature a cusp at the origin $x = 0$. A close similarity between the two response functions for $\delta = \pi^{-1/2}$ was discussed and illustrated in Ref. [43].

B. Comparison with dipolar model

We have checked that all the above results are qualitatively preserved when the calculations are performed with physically more realistic kernel function (16). The VA results follow from the same effective Lagrangian but with the last term in Eq. (5) replaced by $gN\delta^3/(8\pi)\tilde{F}(a,\delta)$, where $\tilde{F}(a,\delta)$ denotes a complicated function, involving modified Bessel functions and omitted here for brevity.

In spite of bulky analytical expressions, it is possible to solve the VA equations numerically and compare the results with the governing GPE, involving the kernel function (16). In Fig. 8, we show the stationary wave profile of a three-soliton molecule and the potential curve, obtained using Eq. (16).

In Fig. 9, we illustrate the GPE and VA dynamics of a three-soliton molecule under time periodic modulation of the dipolar interaction. Comparison of Figs. 1 and 2, obtained using the Gaussian kernel Eq. (2), with Figs. 8 and 9, constructed using the kernel Eq. (16), shows their qualitative similarity. The same is true for other properties of a soliton molecule discussed in Secs. III and IV. Thus, we conclude that the above phenomena should be observable in a quasi-1D dipolar BEC.

Second, in our numerical simulations we have neglected by contact interactions, assuming the dipolar interactions to be dominant. However, these results survive also in the presence of contact atomic interactions. The dimensionless quantity, characterizing the strength of dipole-dipole interactions with respect to contact interactions, is given by

$$\varepsilon = \frac{\mu_0\mu^2 m}{12\pi\hbar^2 a_s}, \tag{17}$$

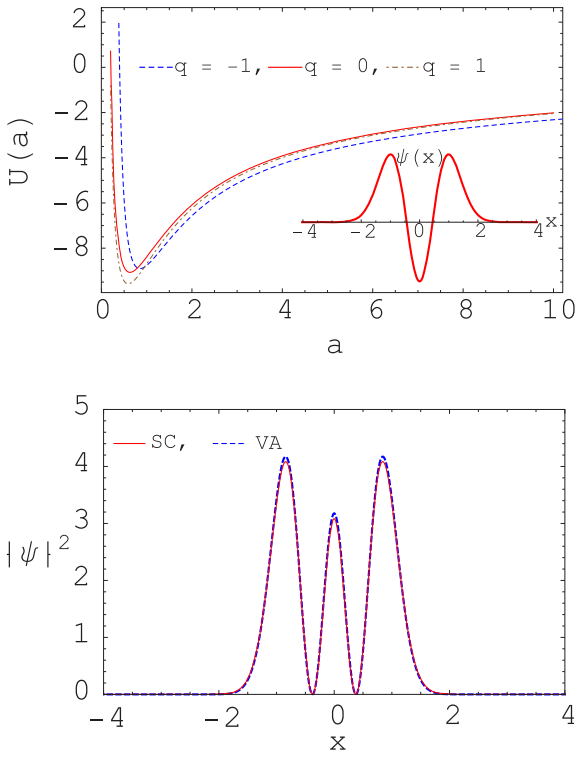

FIG. 8. Top panel: The potential $U(a)$ for $g = 10$ and $q = 0, \pm 1$, constructed from VA equations, using the kernel function (16). The inset shows the wave function of a three-soliton molecule for $q = 0$. Bottom panel: The modulo square of the wave function, according to VA for parameter values $N = 6$, $\delta = 4/\sqrt{\pi}$, $q = 0$, $A = 1.78$, and $a = 0.53$. Dashed line represents the stationary profile, constructed using the self-consistent procedure [35].

where μ_0 is the permeability of vacuum, μ, m are the magnetic dipole moment and mass of the atom, respectively, and a_s is the s-wave scattering length, responsible for the contact

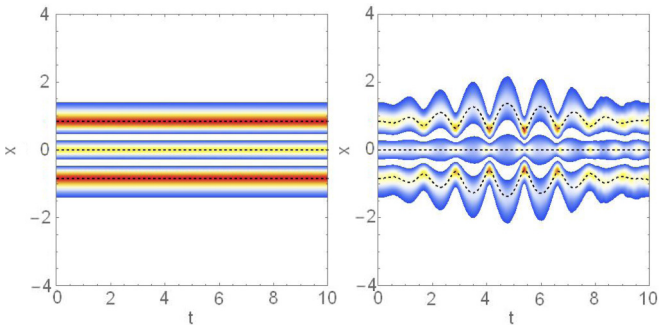

FIG. 9. Left panel: Stable propagation of the three-soliton molecule with parameters predicted by VA for the kernel function (16). The density plot is obtained by numerical solution of the GPE (1). Dashed lines correspond to positions of maxima of the central and lateral solitons $x_m = \pm\sqrt{5/2}\,a(t)$, where the time-dependent parameter $a(t)$ is evaluated from VA. Right panel: Periodic variation of the strength of dipolar interactions $g(t) = g_0[1 + \epsilon\sin(\omega_0 t)]$ at resonant frequency $\omega_0 = 5.619$ gives rise to vibration of lateral solitons with growing amplitude, while the central soliton remains at origin due to the symmetry. Parameter values: $g_0 = 10$, $\epsilon = 0.1$. Other parameters are similar to Fig. 8.

interactions and expressed in units of Bohr radius a_0. For electric dipole moments, the formula is similar with replacement $\mu_0\mu^2 \to d^2/\epsilon_0$, where d is the electric dipole moment of atoms and ϵ_0 is the permittivity of vacuum. For the family of dipolar BECs, the estimates are as follows: ^{52}Cr, $\mu = 6\,\mu_B$, $a_s = 16\,a_0$, $\varepsilon_{\rm Cr} = 0.16$; ^{164}Dy, $\mu = 10\,\mu_B$, $a_s = 92\,a_0$, $\varepsilon_{\rm Dy} = 1.4$; and ^{168}Er, $\mu = 7\,\mu_B$, $a_s = 60\,a_0$, $\varepsilon_{\rm Er} = 0.4$. Comparing these values of ε with that of the nondipolar condensate ^{87}Rb ($\mu = 1.0\,\mu_B$, $a_s = 0.7\,a_0$, $\varepsilon_{\rm Rb} = 0.007$), we conclude that the dipolar interactions in Cr, Dy, and Er dominantly contribute to scattering properties of BEC. On the other hand, when the objective is to observe the dipolar effects clearly, the contact interactions can be reduced to zero by a magnetic or optical Feshbach resonance technique [38]. In the experiments, the setting considered in this paper could be implemented by applying suitable optical or magnetic fields to create weakly stretching potentials for the soliton molecule.

In conclusion, we have introduced a three-soliton molecule which can exist in BEC with nonlocal atomic interactions confined to quasi-1D traps. The stationary wave form, potential of intersoliton interaction, the bond length, and some other characteristic parameters of the three-soliton molecule are obtained using the variational approach and confirmed by numerical simulations of the nonlocal GPE. To explore the normal mode dynamics of the three-soliton molecule, we imposed constant and nonuniform chirping of the ground-state wave function and used them as initial conditions in numerical simulations of the nonlocal GPE. We have shown that, contrary to usual triatomic molecules, the frequency of the asymmetric mode of a three-soliton molecule is always smaller than the one of the symmetric mode. Comparison of the frequencies of small amplitude oscillations of individual solitons, obtained from numerical solution of the nonlocal GPE, showed a good agreement with the predictions of the developed model. The results of the present work can be of interest, e.g., in studies of oscillations of the dipolar BEC over the surface trap, made of a superconductor material [44,45]. Normal modes of soliton molecules can be experimentally measured also in dispersion-managed optical fibers, where three-soliton molecules are already produced [4,5].

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